ROM2F/2004/35

Rationality of the Anomalous Dimensions in $\mathcal{N}$=4 SYM theory

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Abstract

We reconsider the general constraints on the perturbative anomalous dimensions in conformal invariant QFT and in particular in $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. We show that all the perturbative corrections to the anomalous dimension of a renormalized gauge invariant local operator can be written as polynomials in its one loop anomalous dimension. In the $\mathcal{N} = 4$ SYM theory the coefficients of these polynomials are rational functions of the number of colours N.

1 Introduction and summary of the results

In a conformal invariant quantum field theory (CFT) in which the conformal operator product expansion (OPE) holds, all n-point correlation functions are completely determined if the following two ingredients are known:

- the spectrum of the conformal (scale) dimensions $\Delta(g^2)$, or equivalently the anomalous dimensions $\gamma(g^2) = \Delta(g^2) - \Delta_0$ of all the operators;

- the OPE structure constants and the normalizations of the 3-point functions of the operators[1]. The structure constants determine in particular also the "Fusion rules" *i.e.* the (conformal families of) operators that can appear in the product of two given operators.

Following the conjectured AdS/CFT correspondence [1], CFT and in particular $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory has been extensively studied in the last several years. Considerable progress has been achieved both in understanding the general structure of the theory and in the determination of the spectrum of the anomalous dimensions. In particular the complicated structure of $SU(2,2|4)$ supermultiplets has been understood [2, 3] and a classification of the various shortening conditions was obtained [4, 5, 6]. Non-renormalization theorems for various types of functions have been proven: for 1/2 BPS operators [7], as well as extremal [8] and next-to-extremal correlators [9]. The implications of the superconformal Ward identities have also been investigated in detail [10, 11, 12].

Explicit calculations of 2-, 3- and 4-point correlation functions (mostly of protected 1/2 BPS operators) have been performed up to order g^4 [13, 14, 15, 16, 17, 18, 19, 20, 21]. The simplest 4-point correlation functions involving non-protected operators (the lowest component of the Konishi supermultiplet) have been computed in [22, 23]. All these computations confirmed (at least up to order g^4) the predicted finiteness of the correlation functions of gauge invariant operators, and the resummation of the perturbative logarithms in powers, but they also demonstrated a complicated mixing pattern for the operators in the theory.

The spectrum of anomalous dimensions and the related mixing problem has been investigated by (essentially) three different approaches:

- by explicit perturbative calculations of 2-point functions at order g^2, g^4 and recently for the Konishi multiplet also at order g^6 [24, 23, 25, 26, 27, 28, 29, 30, 31, 32];

- by OPE analysis of 4-point functions [14, 15, 17, 18, 33, 11, 22, 23, 21];

- by explicit diagonalization of the action of the Dilation generator [34, 35, 36, 37, 38, 39]. This approach, which (in principle) gives all anomalous dimensions at order g^2 [36], combined with the integrability assumption in the planar limit ($N \to \infty$) predicts the

[1] These two quantities are related by the normalizations of the 2-point functions of the operators, which are arbitrary. Only the ratio of the square of the normalization of the 3-point function and the product of the normalizations of the 2-point functions has a invariant meaning.

values of some anomalous dimensions up to order g^{12}.

To summarize, we have a lot of data at least for the first several perturbative corrections to the naive scale dimensions of many operators. Less is known, however, for the general structure of these perturbative corrections.

In this paper, rather than performing explicit perturbative calculations, we shall reconsider the general constraints on the perturbative anomalous dimensions in any CFT and in particular in $\mathcal{N}=4$ SYM with gauge group $SU(N)$. The main result of our analysis are two properties, which we call *Universality* and *Rationality*. To formulate them we need the following definition: Let $\{\mathcal{O}_i\}$, $i=1,\ldots,d$, be a complete set of gauge invariant bare operators which have the same tree level quantum numbers and thus can mix among themselves. The counterpart of this set, after resolving the mixing in the interacting CFT, are exactly d renormalized operators $\widehat{\mathcal{O}}_k$, $k=1,\ldots,d$, with well defined anomalous dimensions. We define a *Class of renormalized operators* as the set $\{\widehat{\mathcal{O}}_k\}$ which contains all the operators $\widehat{\mathcal{O}}_k$, $k=1,\ldots,d$, corresponding to the same set of bare operators $\{\mathcal{O}_i\}$. It follows that each renormalized operator belongs to exactly one class[2]. We shall prove:

Universality : In a finite CFT, the anomalous dimensions of any renormalized operator $\widehat{\mathcal{O}}_k$ can be written as a polynomial

$$\gamma_k(g^2) = \sum_{\ell=0}^{d-1} w_\ell\,(g^2)\left(\gamma_k^{(1)}\right)^\ell . \tag{1}$$

Here $\gamma_k(g^2)$ is the complete (perturbative) anomalous dimension of the operator $\widehat{\mathcal{O}}_k$, $\gamma_k^{(1)}$ is its one loop (order g^2) anomalous dimension, and d is the dimension of the class of renormalized operators $\{\widehat{\mathcal{O}}_k\}$ to which $\widehat{\mathcal{O}}_k$ belongs. The functions $w_\ell\,(g^2)$ are universal, *i.e.* they are the same for all the operators in a given class of renormalized operators. For two different classes of renormalized operators, these functions in general will be different.

Rationality : In $\mathcal{N}=4$ SYM theory with gauge group $SU(N)$ the coefficients $w_\ell\,(g^2)$ in eq. (1) will depend also on the number of colours N. The functions $w_\ell\,(g^2,N)$ have a power series expansion in g^2 with coefficients that are rational functions (*i.e.* ratio of polynomials) of the number of colours N. In other words for all p the order g^{2p} anomalous dimension of the operator $\widehat{\mathcal{O}}_k$, $\gamma_k^{(p)}$, can be written as a polynomial in $\gamma_k^{(1)}$ with universal coefficients, $r_\ell^{(p)}(N)$, rational in N

$$\gamma_k^{(p)} = \sum_{\ell=0}^{d-1} r_\ell^{(p)}(N)\,\left(\gamma_k^{(1)}\right)^\ell . \tag{2}$$

[2]The partition of the space of operators in classes should not be confused with the supermultiplet structure in $\mathcal{N}=4$ SYM. The members of the same class of renormalized operators can belong to different supermultiplets. Moreover they can be superprimary as well as superdescendants.

Similar considerations apply for the normalizations of the n-point functions in the theory, which also can be written as polynomials in the one loop anomalous dimensions $\gamma^{(1)}$ of the respective operators.

As an explicit illustration of these considerations we present the rational (in the sense of eq. (2)) representation of the order g^4 anomalous dimensions of the scalar operators of naive scale dimension $\Delta_0 = 4$ in the $\mathbf{20'}$ representation of the $SU(4)$ R-symmetry in $\mathcal{N} = 4$ SYM theory.

2 Anomalous dimensions in conformal field theory

We shall first recall some relevant general properties of the 2-point functions in a conformal invariant theory [31]. Let the scalar[3] operators $\widetilde{\mathcal{O}}_i(x,\epsilon)$, with $i = 1,\dots,d$, be a set of bare regularized (by point-splitting with separation ϵ) operators which can mix among themselves, hence they have the same naive dimension Δ_0 [4]. The tilde in $\widetilde{\mathcal{O}}_i(x,\epsilon)$ denotes that these operators are properly subtracted, as discussed in [31]. We want to resolve the mixing problem and find the corresponding anomalous dimensions. The result of the perturbation theory calculation will have the form[5]

$$\langle \widetilde{\mathcal{O}}_i(x,\epsilon)\ \widetilde{\mathcal{O}}^\dagger_j(y,\epsilon)\rangle = f_{ij}\left(\frac{\epsilon^2}{(x-y)^2}, g\right)\ \frac{1}{[(x-y)^2]^{\Delta_0}}\,, \tag{3}$$

where f_{ij} is the non-vanishing (either divergent or finite) in the limit $\epsilon \to 0$ part of the correlator. It is a hermitian matrix depending on the operator basis we have chosen. Actually, since complex operators come in pairs with the same anomalous dimension we can always choose a basis in which f_{ij} is real and symmetric.

The renormalized operators from the class $\{\widehat{\mathcal{O}}_k\}$, $k = 1,\dots,d$, which have well defined anomalous dimensions $\gamma_k(g^2)$ are linear combinations of the bare operators $\widetilde{\mathcal{O}}_j$

$$\widehat{\mathcal{O}}_k(x,\mu) = \sum_{j=1}^{d} Z_{kj}(\epsilon^2\mu^2, g)\ \widetilde{\mathcal{O}}_j(x,\epsilon)\,, \tag{4}$$

where the auxiliary scale μ is the subtraction point, and Z is the invertible mixing matrix. In the basis in which f_{ij} is real and symmetric Z has the property $Z^\dagger = Z^T$, where the

[3] We choose scalars just for simplicity, the case of operators of arbitrary spin is not essentially different.

[4] This, together with the equality of the spins of the operators, is only a necessary condition for mixing, in particular cases the operators have to satisfy more conditions, *e.g.* in the case of $\mathcal{N} = 4$ SYM they have to belong to the same $SU(4)$ representation.

[5] Although to simplify the formulae we shall write all the relations as depending on only one coupling constant, g, the generalization to the case of several perturbative coupling constants is straightforward.

superscript T means transposition. Scale invariance implies that the two-point functions of $\widehat{\mathcal{O}}_k(x,\mu)$ have the form

$$\langle \widehat{\mathcal{O}}_k(x,\mu)\ \widehat{\mathcal{O}}^\dagger_\ell(y,\mu)\rangle = \frac{\delta_{k\ell}\ M_k(g)}{[(x-y)^2]^{\Delta_0}[(x-y)^2\mu^2]^{\gamma_k(g^2)}}\,, \tag{5}$$

where we have separated the dependence on the naive and the anomalous dimension. $M_k(g)$ are the *finite* normalizations of the 2-point functions of the renormalized operators [6].

Let us stress that, while f_{ij}, Z_{kj} and M_k can in general depend on both even and odd powers of the coupling constant g, the physical anomalous dimensions, γ_k, can only be functions of g^2. Compatibility among equations (3), (4) and (5) implies the matrix equation

$$Z(\epsilon^2\mu^2,g)\ f\left(\frac{\epsilon^2}{(x-y)^2},g\right)\ Z^\dagger(\epsilon^2\mu^2,g) = \left[(x-y)^2\mu^2\right]^{-\Gamma(g^2)}\ M(g)\,, \tag{6}$$

where $\Gamma(g^2)$ and $M(g)$ are the diagonal matrices of anomalous dimensions and normalizations of the 2-point functions respectively. Unitarity implies that $M_k(g)$ are all positive. Thus, since there exists a basis in which both f and Z are real, the anomalous dimensions are also all real.

We write eq. (6) in two special cases, namely

- for $\epsilon^2\mu^2=1$ and $(x-y)^2\mu^2 = 1/\rho$ which yields

$$Z(1,g)\ f(\rho,g)\ Z^\dagger(1,g) = \rho^{\Gamma(g^2)}\ M(g)\,, \tag{7}$$

- for $\epsilon^2\mu^2=\rho$ and $(x-y)^2\mu^2 = 1$ which yields

$$Z(\rho,g)\ f(\rho,g)\ Z^\dagger(\rho,g) = M(g)\ . \tag{8}$$

It follows that if $Z(1,g)$ is a solution of eq. (7), then

$$Z(\rho,g) = \rho^{-\frac{1}{2}\Gamma(g^2)}\ Z(1,g) \tag{9}$$

is a solution of eq. (8). The last relation has a simple intuitive meaning, one first defines by means of $Z(1,g)$ the operators with well defined scale dimension, then renormalizes them by the factor $(\epsilon^2\mu^2)^{-\frac{1}{2}\Gamma(g^2)}$. Thus the ρ dependence in $Z(\rho,g)$ factorizes and we have to solve only eq. (7) for the unknown $Z(1,g)$ and $\Gamma(g^2)$ for a given function $f(\rho,g)$ and a choice of $M(g)$.

[6] Although the 2-point functions can be always normalized to one, as we shall see it is convenient to relax this constraint, so we shall leave these normalizations as free parameters for the moment.

Rather than trying to solve explicitly these relations (like in [31]), in this paper we shall study the implications of their general structure. In order to simplify the notation we shall denote $Z(1, g)$ by $Z(g)$. Then equation (7) can be written as

$$\sum_{k=1}^{d} Z_{ik}^{-1}(g)\, Z_{jk}^{-1}(g)\, M_k(g)\, \rho^{\gamma_k(g^2)} = f_{ij}(\rho, g)\,. \tag{10}$$

Introducing the definition

$$A_{ij}^k(g) = Z_{ik}^{-1}(g)\, Z_{jk}^{-1}(g)\, M_k(g) \tag{11}$$

and expanding both sides in power series in $\ln(\rho)$

$$f_{ij}(\rho, g) = \sum_n \frac{(\ln(\rho))^n}{n!} F_{ij}^n(g)\ , \qquad \rho^{\gamma_k(g^2)} = \sum_n \frac{(\ln(\rho))^n}{n!} \left(\gamma_k(g^2)\right)^n\ , \tag{12}$$

we get for every $n \geq 0$

$$\sum_{k=1}^{d} A_{ij}^k(g)\ \left(\gamma_k(g^2)\right)^n = F_{ij}^n(g)\,. \tag{13}$$

Note that although the range of all three indices i, j and k in the above equation is the same (from 1 to d), the first two (i and j) label the bare regularized operators, while the last one, k, labels the renormalized operators.

Let us now specialize to the case of $\mathcal{N} = 4$ SYM with a gauge group $SU(N)$. All the quantities in the theory will depend also on the number of colours N. If we take this into account then equation (13) becomes

$$\sum_{k=1}^{d} A_{ij}^k(g, N)\ \left(\gamma_k(g^2, N)\right)^n = F_{ij}^n(g, N) \tag{14}$$

for every $n \geq 0$.

The properties of the renormalized operators are independent of the choice of the basis of bare regularized operators. We shall use this freedom, and choose bare operators which do not contain explicit non-rational dependence on N. Such a choice indeed always exists, for example the pure colour traces which do not contain any explicit N dependence satisfy this requirement. With such a choice of basis of bare operators, since the colour contractions can produce only factors rational in N, the rhs of eq. (14) can be expanded in power series in g as

$$F_{ij}^n(g, N) = \sum_{p=0}^{\infty} g^{2n+p}\ R_{ij}^{n\,(p)}(N)\,, \tag{15}$$

where $R^{n\,(p)}_{ij}(N)$ are all rational functions (ratios of polynomials) of N. Note that even if all $R^{n\,(p)}_{ij}$ are rational, F^n_{ij} can still be non-rational functions of N due to the infinite sum in eq. (15).

Expanding in power series in g also the lhs of eq. (14) (and in order to simplify the notation again suppressing the N dependence) and putting

$$\gamma_k(g^2) = g^2\gamma_k^{(1)} + g^4\gamma_k^{(2)} + g^6\gamma_k^{(3)} + \dots\,, \tag{16}$$

$$A^k_{ij}(g) = A^{k(0)}_{ij} + gA^{k(1)}_{ij} + g^2A^{k(2)}_{ij} + g^3A^{k(3)}_{ij} + g^4A^{k(4)}_{ij} + \dots\,, \tag{17}$$

we get

$$\begin{aligned}
& g^{2n}\sum_k \left(\gamma_k^{(1)}\right)^n A^{k(0)}_{ij} + \\
& g^{2n+1}\sum_k \left(\gamma_k^{(1)}\right)^n A^{k(1)}_{ij} + \\
& g^{2n+2}\sum_k \left(\gamma_k^{(1)}\right)^{n-1} \left(n\,A^{k(0)}_{ij}\,\gamma_k^{(2)} + A^{k(2)}_{ij}\,\gamma_k^{(1)}\right) + \\
& g^{2n+3}\sum_k \left(\gamma_k^{(1)}\right)^{n-1} \left(nA^{k(1)}_{ij}\gamma_k^{(2)} + A^{k(3)}_{ij}\gamma_k^{(1)}\right) + \\
& g^{2n+4}\sum_k \left(\gamma_k^{(1)}\right)^{n-2} \left(\frac{n(n-1)}{2}A^{k(0)}_{ij}\left(\gamma_k^{(2)}\right)^2 + n\,A^{k(0)}_{ij}\,\gamma_k^{(3)}\gamma_k^{(1)} + \right. \\
& \qquad\qquad \left. n\,A^{k(2)}_{ij}\,\gamma_k^{(2)}\,\gamma_k^{(1)} + A^{k(4)}_{ij}\left(\gamma_k^{(1)}\right)^2\right) + \dots\,.
\end{aligned} \tag{18}$$

Note that since the terms involving $A^{k(p)}_{ij}$ with p even/odd are always multiplied by an even/odd power of g the corresponding equations are decoupled. So let us first consider the system of equations involving only $A^{k(p)}_{ij}$ with p even. Comparing eq. (15) and eq. (18) we find

$$\sum_{k=1}^d \left(\gamma_k^{(1)}\right)^n A^{k(0)}_{ij} = R^{n(0)}_{ij}\,, \tag{19}$$

$$\sum_{k=1}^d \left(\gamma_k^{(1)}\right)^{n-1} \left(n\,A^{k(0)}_{ij}\,\gamma_k^{(2)} + A^{k(2)}_{ij}\,\gamma_k^{(1)}\right) = R^{n(2)}_{ij}\,, \tag{20}$$

$$\begin{aligned}
& \sum_{k=1}^d \left(\gamma_k^{(1)}\right)^{n-2} \left(\frac{n(n-1)}{2}A^{k(0)}_{ij}\left(\gamma_k^{(2)}\right)^2 + n\,A^{k(2)}_{ij}\,\gamma_k^{(2)}\,\gamma_k^{(1)} + \right. \\
& \qquad\qquad \left. n\,A^{k(0)}_{ij}\,\gamma_k^{(3)}\,\gamma_k^{(1)} + A^{k(4)}_{ij}\left(\gamma_k^{(1)}\right)^2\right) = R^{n(4)}_{ij}\,,
\end{aligned} \tag{21}$$

and so forth.

In order to proceed we shall make an important assumption, namely that the order g^2 corrections to the anomalous dimensions, $\gamma_k^{(1)}$, of the operators in the same class are non-degenerate, *i.e.* they are all different. We shall return later to the more complicated degenerate case.

We shall first analyze the system of equations (19) which involves only the first corrections to the anomalous dimensions of the operators $\gamma_k^{(1)}$. Let us define

$$\begin{aligned} S_0 &= 1\,, \\ S_1 &= -\sum_{k_1} \gamma_{k_1}^{(1)}\,, \\ S_2 &= \sum_{k_1<k_2} \gamma_{k_1}^{(1)}\,\gamma_{k_2}^{(1)}\,, \\ S_2 &= -\sum_{k_1<k_2<k_3} \gamma_{k_1}^{(1)}\,\gamma_{k_2}^{(1)}\,\gamma_{k_3}^{(1)}\,, \\ & \dots \\ S_d &= (-1)^d\,\gamma_1^{(1)}\,\gamma_2^{(1)}\dots\gamma_d^{(1)}\,. \end{aligned} \tag{22}$$

Then the system of equations (19) implies

$$\sum_{\ell=0}^{d} S_\ell\; R_{ij}^{n+d-\ell\;(0)} = 0\,, \tag{23}$$

for every $n \geq 0$. From this it follows that

- all S_ℓ are rational functions of N;
- all $\gamma_k^{(1)}$ are roots of a degree d polynomial equation with coefficients rational in N

$$P_d(\gamma_k^{(1)}) = \sum_{\ell=0}^{d} S_\ell\,\left(\gamma_k^{(1)}\right)^{d-\ell} = 0\,; \tag{24}$$

- an arbitrary (fixed) power of $\gamma_k^{(1)}$ can be written as a polynomial of degree not higher than $d-1$ in $\gamma_k^{(1)}$ with coefficients rational in N. Indeed with the help of eq. (24) we can express $(\gamma_k^{(1)})^d$ and all the higher powers in such a form.

We shall need also the following properties:

Property 1. Any polynomial $Q_s(\gamma_1^{(1)}, \gamma_2^{(1)}, \dots, \gamma_d^{(1)})$ with coefficients rational in N, totaly symmetric in its d arguments, can be written as a rational function only of N.

Indeed, since Q_s is totally symmetric, it is invariant under the permutation of its arguments. Hence it can always be expressed as a polynomial (with coefficients rational

in N) in the basis of permutation invariants $S_1, \ldots, S_d$, defined in eq. (22) (which are also rational functions of N).

Property 2. If Q is an arbitrary rational function (ratio of polynomials) of $\gamma_k^{(1)}$, $k = 1, \ldots, d$, with coefficients rational in N, then it can be written in an equivalent form as a polynomial in $\gamma_k^{(1)}$, $k = 1, \ldots, d$, with coefficients rational in N.

This follows by noting that by an appropriate multiplication with a polynomial in $\gamma_k^{(1)}$ we can complete the polynomial in the denominator of Q to a permutation invariant, which due to Property 1. is a rational function only of N.

Property 3. If $Q_1(\gamma_1^{(1)}; \gamma_2^{(1)}, \ldots, \gamma_d^{(1)})$ is a rational function of all its arguments, symmetric in $\gamma_2^{(1)}, \ldots, \gamma_d^{(1)}$, with coefficients rational in N, then it can be written as a polynomial in $\gamma_1^{(1)}$ (of degree not higher than $d-1$) with coefficients rational in N.

First, the denominator can be eliminated as in the previous case without changing the symmetry of Q_1. Second, due to the symmetry in $\gamma_2^{(1)}, \ldots, \gamma_d^{(1)}$, we can write the resulting polynomial in terms of the permutation invariants of $\gamma_2^{(1)}, \ldots, \gamma_d^{(1)}$, which in turn can be expressed in terms of S_ℓ of eq. (22) and $\gamma_1^{(1)}$. In particular one has

$$
\begin{aligned}
-\sum_{2\leq k} \gamma_k^{(1)} &= S_1 + \gamma_1^{(1)}\,, && (25)\\
\sum_{2\leq k_1<k_2} \gamma_{k_1}^{(1)}\gamma_{k_2}^{(1)} &= S_2 - \gamma_1^{(1)} \sum_{2\leq k} \gamma_k^{(1)} = S_2 + \gamma_1^{(1)}\left(S_1 + \gamma_1^{(1)}\right)\,, && (26)
\end{aligned}
$$

and so forth.

After these rather long preliminaries let us return to the system of eq. (19). We can solve it for $A_{ij}^{k(0)}$ as functions of $\gamma_1^{(1)}, \gamma_2^{(1)}, \ldots, \gamma_d^{(1)}$. The solution

$$
A_{ij}^{k(0)} = A_{ij}^{k(0)}\left(\gamma_k^{(1)}; \{\gamma^{(1)}\}\diagdown\gamma_k^{(1)}\right) \tag{27}
$$

is a rational function, symmetric in the $d-1$ variables $\{\gamma^{(1)}\}\diagdown\gamma_k^{(1)}$. Thus by Property 3. we can write it in an equivalent form as

$$
A_{ij}^{k(0)} = \sum_{\ell=0}^{d-1} a_{ij}^{(0)\ell}(N)\left(\gamma_k^{(1)}\right)^\ell\,, \tag{28}
$$

where $a_{ij}^{(0)\ell}$ are rational functions of N. Note that they do not depend on k, *i.e.* on the particular operator in the class of renormalized operators.

Let us now proceed to the more complicated system of eq. (20), which contains both the first and the second corrections to the anomalous dimensions of the operators $\gamma_k^{(1)}$ and

$\gamma_k^{(2)}$. We can rewrite it in the form

$$\sum_k \left(\gamma_k^{(1)}\right)^n A_{ij}^{k(2)} = R_{ij}^{n(2)} - n \sum_k A_{ij}^{k(0)} \gamma_k^{(2)} \left(\gamma_k^{(1)}\right)^{n-1} = W_{ij}^n \ . \tag{29}$$

Repeating the derivation of eq. (23) from eq. (19) we can exclude all $A_{ij}^{k(2)}$ and obtain a system of equations, for every $n \geq 1$, involving only S_ℓ and W_{ij}^n

$$\sum_{\ell=0}^{d} S_\ell \, W_{ij}^{n+d-\ell} = 0 \ . \tag{30}$$

This is a linear system for $\gamma_k^{(2)}$, whose solution (if we use also eq. (28), and the assumed non-degeneracy of all $\gamma_k^{(1)}$) has the form

$$\gamma_k^{(2)} = \gamma_k^{(2)} \left(\gamma_k^{(1)}; \{\gamma^{(1)}\} \diagdown \gamma_k^{(1)}\right) , \tag{31}$$

which implies for every $k = 1, \ldots, d$

$$\gamma_k^{(2)} = \sum_{\ell=0}^{d-1} r_\ell^{(2)}(N) \left(\gamma_k^{(1)}\right)^\ell \ . \tag{32}$$

Here $r_\ell^{(2)}(N)$ are rational functions of N, which do not depend on k. Let us stress that although for demonstrating this we made use of a particular basis of bare operators (*e.g.* pure colour traces), the rationality property is independent of any choice of basis, since the anomalous dimensions are invariant properties of the renormalized operators.

Neglecting the N-dependence, which is particular for the $\mathcal{N} = 4$ SYM theory, equation (32) implies that in any CFT the order g^4 correction to the anomalous dimension of every operator can always be written as a polynomial in the order g^2 correction to the anomalous dimension of the same operator. The coefficients of the polynomial are the same for different operators belonging to the same class of renormalized operators.

Inserting eq. (32) into eq. (29) and solving for $A_{ij}^{k(2)}$ we find also

$$A_{ij}^{k(2)} = \sum_{\ell=0}^{d-1} a_{ij}^{(2)\ell}(N) \left(\gamma_k^{(1)}\right)^\ell , \tag{33}$$

where $a_{ij}^{(2)\ell}$ are again rational functions of N, which do not depend on k.

We are now ready to solve the system of eq. (21). Substituting in it eqs. (28), (32) and (33) and bringing the terms in the first line to the rhs, we get

$$\sum_k \left(\gamma_k^{(1)}\right)^{n-1} \left(n \, A_{ij}^{k(0)} \gamma_k^{(3)} + A_{ij}^{k(4)} \gamma_k^{(1)}\right) = \widetilde{R}_{ij}^{n(4)} , \tag{34}$$

where $\widetilde{R}_{ij}^{n(4)}$ are again rational functions of N, because they differ from $R_{ij}^{n(4)}$ by a totaly symmetric in $\gamma_k^{(1)}$ $(k = 1, \ldots, d)$ expression. This system is similar to the system in eq. (20), so it is immediate to write its solution, namely

$$\gamma_k^{(3)} = \sum_{\ell=0}^{d-1} r_\ell^{(3)}(N) \left(\gamma_k^{(1)}\right)^\ell , \tag{35}$$

and

$$A_{ij}^{k(4)} = \sum_{\ell=0}^{d-1} a_{ij}^{(4)\ell}(N) \left(\gamma_k^{(1)}\right)^\ell , \tag{36}$$

with $r_\ell^{(3)}$ and $a_{ij}^{(4)\ell}$ rational functions of N.

Proceeding in the same way, and taking into account also the equations for $A_{ij}^{k(p)}$ with p odd implied by eqs. (15) and (18), we derive

$$\gamma_k^{(p)} = \sum_{\ell=0}^{d-1} r_\ell^{(p)}(N) \left(\gamma_k^{(1)}\right)^\ell \tag{37}$$

and

$$A_{ij}^{k(p)} = \sum_{\ell=0}^{d-1} a_{ij}^{(p)\ell}(N) \left(\gamma_k^{(1)}\right)^\ell \tag{38}$$

for every i, j, k and p with coefficients $r_\ell^{(p)}$ and $a_{ij}^{(p)\ell}$ rational in N and independent of k.

It follows, from eq. (37), that in any CFT any fixed order correction to (and hence also the total perturbative) anomalous dimension of every operator can always be written as a polynomial in the order g^2 correction to the anomalous dimension of the same operator

$$\gamma_k(g^2) = \sum_{\ell=0}^{d-1} w_\ell\,(g^2) \left(\gamma_k^{(1)}\right)^\ell . \tag{39}$$

The coefficients of the polynomial, $w_\ell\ (g^2)$, are universal, since they do not depend on k and hence are the same for different operators belonging to the same class of renormalized operators. In particular, in $\mathcal{N} = 4$ SYM, the functions $w_\ell\ (g^2)$ have a power series expansion in g^2 with coefficients $r_\ell^{(p)}$ rational in N.

So far we considered the generic case when the order g^2 anomalous dimensions are just roots of a degree d polynomial with coefficients rational in N (see eq. (24)). It often happens that this polynomial can be factorized as a product of two (or more) polynomials, with coefficients still rational in N,

$$P_d(\gamma_k^{(1)}) = P_{d_1}(\gamma_k^{(1)})\ P_{d_2}(\gamma_k^{(1)}) = 0\,. \tag{40}$$

In this case the class of operators naturally splits into two subclasses, one containing d_1, the other $d_2 = d - d_1$ operators. All the statements about the permutation invariants and their N dependence are valid separately for each of the two subclasses, so we can write eq. (39) in the following equivalent form

$$\begin{aligned} \gamma_{k_1}(g^2) &= \sum_{\ell=0}^{d_1-1} u_\ell \; (g^2) \left(\gamma_{k_1}^{(1)}\right)^\ell , \\ \gamma_{k_2}(g^2) &= \sum_{\ell=0}^{d_2-1} v_\ell \; (g^2) \left(\gamma_{k_2}^{(1)}\right)^\ell , \end{aligned} \tag{41}$$

where $\gamma_{k_j}^{(1)}$, $j = 1, 2$ are the roots of $P_{d_j}(\gamma_{k_j}^{(1)}) = 0$ respectively. We have decreased the degree of the polynomials in this expressions, but at the price of using different functions u_ℓ and v_ℓ for the two subclasses of operators. Note that the number of variables is the same for both representations since $d = d_1 + d_2$. Moreover, if the functions $w_\ell \; (g^2)$ admit a power series expansion in g^2 with coefficients rational in N, so will the functions u_ℓ and v_ℓ. As a consequence of eqs. (40) and (41), $\gamma_{k_j}^{(p)}$ at all orders in perturbation theory will be given by the roots of a polynomial of degree not higher than d_j. Hence the factorization of the equation for $\gamma^{(1)}$ implies the factorization of the equation for all $\gamma^{(p)}$. Thus the dimensions of the closed (in the sense of eq. (40)) one loop subclasses are preserved to higher loops. The generalization to more than two factors in eq. (40) is straightforward.

The factorization of the polynomials defining $\gamma^{(1)}$ (like in eq. (40)) is of particular importance in $\mathcal{N} = 4$ SYM theory. The reason is that, on the one hand the operators in $\mathcal{N} = 4$ SYM are organized in large supermultiplets with typical number of components of the order of 2^{16}, all with the same anomalous dimension $\gamma(g^2)$. On the other hand, the relevant quantum numbers for the resolution of the mixing problem considered in this paper are the naive scale dimension Δ_0 and the spin s of the operators and the $SU(4)$ representation to which they belong. Hence, in general, the set of d operators we start with may contain both operators which belong to *similar* supermultiplets (with the same quantum numbers of the naive, for $g = 0$, lowest component and the same number of components) and to *essentially different* supermultiplets (with different quantum numbers of the naive lowest components and/or different number of components). It turns out that the factorization properties of the polynomial P_d defining $\gamma^{(1)}$ describe exactly this structure. To be more precise, if the order g^2 corrections to the anomalous dimensions of two operators are roots of a non-factorizable (with coefficients rational in N) polynomial, then these two operators belong to similar supermultiplets. The proof follows by noting that if two supermultiplets are essentially different then there exists at least one component which belongs to only one, say the first, of them. Let us consider the class of renormalized operators corresponding to such a component. It follows that this class

will contain an operator belonging to the first supermultiplet, but will not contain an operator belonging to the second supermultiplet. This contradicts our assumption that the polynomial is non-factorizable, and hence has as roots always both anomalous dimensions. In other words, if two operators belong to essentially different supermultiplets then their order g^2 anomalous dimensions will be roots of different factors of the polynomial in eq. (40). Hence we can identify the subclasses of renormalized operators with the families of essentially different supermultiplets. The functions, $u_\ell\ (g^2)$ and $v_\ell\ (g^2)$, in eqs. (41) will be universal for all the similar supermultiplets within each family. Whether the inverse is also true, *i.e.* if the factorization of the polynomial implies that the respective supermultiplets are essentially different is an open challenging problem.

An important particular case of factorization is when the order g^2 anomalous dimension of some operator $\widehat{\mathcal{O}}_k$, $\gamma_k^{(1)}$, is a rational function of N (this corresponds to some $d_j = 1$), as *e.g.* for the components of the Konishi supermultiplet $\mathcal{K}$ in the $\mathcal{N} = 4$ SYM theory. Then it follows that for such an operator all perturbative corrections, $\gamma_k^{(p)}$, will be rational functions of N. Still the complete anomalous dimension, being an infinite series, may not share this property.

Before proceeding, let us briefly comment also on the degeneracy problem. So far we have assumed that the order g^2 corrections to the anomalous dimensions of the operators, $\gamma_k^{(1)}$, are non-degenerate, *i.e.* they are all different. This assumption is essential in deriving the unique representations for the higher order anomalous dimensions (see *e.g.* eq. (37)). Indeed, if two order g^2 anomalous dimensions, say $\gamma_{k_1}^{(1)}$ and $\gamma_{k_2}^{(1)}$, are equal then the determinant of the coefficients in eqs. (19) - (21) is zero. There are two distinct types of degeneracy. On the one hand there are operators which have exactly the same anomalous dimension to all orders in perturbation theory *e.g.* since they belong to the same supermultiplet. There is no way to lift this degeneracy by considering only the 2-point functions and one has to take into account also some 3-point functions to distinguish such operators. On the other hand it may happen[7] that the degeneracy is removed at some higher order in perturbation theory. That is, there exists some q such that $\gamma_{k_1}^{(q)} \neq \gamma_{k_2}^{(q)}$ (more precisely we want that all the q-th order anomalous dimensions are not degenerate). In this case, repeating all the derivations, one can show that all the above formulae remain valid if we replace $\gamma_k^{(1)}$ by $\gamma_k^{(q)}$, hence all anomalous dimensions can be written as polynomials in the non-degenerate order g^{2q} anomalous dimensions $\gamma_k^{(q)}$ (with coefficients independent of k and rational in N). We are convinced that it is more efficient to solve the degeneracy problem case by case, depending on the properties of the particular operators at hand, rather than to develop a general prescription. Thus in the rest of the paper we shall treat again only the non-degenerate case.

[7]Although we are not aware of any explicit example of this kind in $\mathcal{N} = 4$ SYM.

3 Normalizations of the correlation functions

Given the universality and rationality properties of the anomalous dimensions in eqs. (37), (39), a natural question arises: Are there, and if yes under which assumptions, similar formulae for the normalizations of the 2-point functions of the renormalized operators, $M_k(g)$, which enter eq. (5)? In other words, can we normalize the renormalized operators in such a way that

$$M_k(g) = \sum_{\ell=0}^{d-1} m_\ell\,(g,N)\,\left(\gamma_k^{(1)}\right)^\ell\,, \tag{42}$$

where $m_\ell\,(g,N)$ are functions independent of k which admit a power series expansion in g with coefficients rational in N. A sufficient condition is the existence of a rational mixing matrix

$$Z^R_{ki}(g) = \sum_{\ell=0}^{d-1} z_i^\ell\,(g,N)\,\left(\gamma_k^{(1)}\right)^\ell\,, \tag{43}$$

such that $z_i^\ell\,(g,N)$ are independent of k functions which admit a power series expansion in g with coefficients rational in N. Indeed eq. (43), together with equations (11) and (38), imply eq. (42). Let Z^R be a matrix of, rational in N, eigenvectors of $\left(F^0(g)\right)^{-1}(F^1(g))$, such that

$$Z^R(g)\,\left[\left(F^0(g)\right)^{-1}\,F^1(g)\right]\,\left(Z^R(g)\right)^{-1} = \Gamma(g^2)\,, \tag{44}$$

with $F^0(g)$ and $F^1(g)$ defined in eq. (12). Such a matrix always exists, since by expanding in power series in g the eigenvector condition and using the rationality in N of F^0, F^1 and Γ one proves the existence of eigenvectors rational in N. The polynomial form of eq. (43), and hence also of eq. (42), then follows. On the other hand, $Z^R(g)$ diagonalizes simultaneously F^0 and F^1 as required for a mixing matrix

$$\begin{aligned} Z^R(g)\,F^0(g)\,\left(Z^R(g)\right)^\dagger &= M(g)\,, \\ Z^R(g)\,F^1(g)\,\left(Z^R(g)\right)^\dagger &= M(g)\,\Gamma(g^2)\,. \end{aligned} \tag{45}$$

In fact there exists a (unique) unitary mixing matrix Z [31] which satisfies eqs. (45) with $M(g) = 1$, and thus also eq. (44). Then it follows that Z^R and Z can be related by a real diagonal matrix $D(g)$, such that $Z^R(g) = D(g)Z(g)$, which in turn implies eqs. (45) with $M(g) = D^2(g)$. Note that there is the residual freedom of multiplying from the left the matrix $Z^R(g)$ by a diagonal matrix with entries polynomial in $\gamma_k^{(1)}$ and rational in N. Such a transformation preserves the polynomial structure of both eqs. (43) and (42) but modifies the values of $M_k(g)$. However in general it is not possible to get $M(g) = 1$ by such a transformation, hence the standard unitary mixing matrix Z of [31] has entries

non-rational in N. This explains also why we did not impose the standard normalization condition on the 2-point functions in eq. (5).

The generalization to the case of higher point functions is as follows. The normalization, $C_{k_1k_2k_3}$, of the 3-point function $\langle\widehat{\mathcal{O}}_{k_1}(x_1)\ \widehat{\mathcal{O}}_{k_2}(x_2)\ \widehat{\mathcal{O}}_{k_3}(x_3)\rangle$ has the following representation

$$C_{k_1k_2k_3}(g) = \sum_{\ell_1,\ell_2,\ell_3} c^{\ell_1\ell_2\ell_3}(g) \left(\gamma^{(1)}_{k_1}\right)^{\ell_1} \left(\gamma^{(1)}_{k_2}\right)^{\ell_2} \left(\gamma^{(1)}_{k_3}\right)^{\ell_3} . \tag{46}$$

A similar expression holds also for the OPE coefficients defined as

$$C_{k_1k_2}{}^{k_3}(g) = \frac{C_{k_1k_2k_3}(g)}{M_{k_3}(g)} . \tag{47}$$

Both these quantities will depend on the particular normalizations of the 2-point functions of the respective operators, $M_k(g)$, and thus have no invariant meaning. In fact the only physical quantities, that are independent on any normalization choices, are the ratios

$$T_{k_1k_2k_3}(g^2) = \frac{\left(C_{k_1k_2k_3}(g)\right)^2}{M_{k_1}(g)\, M_{k_2}(g)\, M_{k_3}(g)} . \tag{48}$$

Note that, unlike $C_{k_1k_2k_3}(g)$ and $C_{k_1k_2}{}^{k_3}(g)$ which may depend both on even and odd powers of g, $T_{k_1k_2k_3}(g^2)$ is a function of g^2 only. Combining eqs. (46) and (42) we find

$$T_{k_1k_2k_3}(g^2) = \sum_{\ell_1,\ell_2,\ell_3} t^{\ell_1\ell_2\ell_3}(g^2) \left(\gamma^{(1)}_{k_1}\right)^{\ell_1} \left(\gamma^{(1)}_{k_2}\right)^{\ell_2} \left(\gamma^{(1)}_{k_3}\right)^{\ell_3} , \tag{49}$$

where the functions $t^{\ell_1\ell_2\ell_3}(g^2)$ do not depend on the choice of the operators in the three classes of renormalized operators. In particular, in $\mathcal{N}=4$ SYM, the functions $t^{\ell_1\ell_2\ell_3}(g^2, N)$ will have a power series expansion in g^2 with coefficients rational in N. Given the 2- and 3-point functions all higher n-point functions can be obtained by the OPE.

4 Conclusions

In the framework of CFT we derived a representation for all the perturbative corrections to the anomalous dimension of any given gauge invariant operator as polynomials in its one loop anomalous dimension, eq. (1). In the case of $\mathcal{N}=4$ SYM with gauge group $SU(N)$ we have proven that the coefficients of these polynomials are rational functions of the number of colours N, (see eq. (2)).

Since our considerations do not modify the number of unknown functions, it might seem that this is a completely equivalent representation but, as we shall argue, in $\mathcal{N}=4$

SYM this is not the case. The reason is twofold. On the one hand, due to the rationality property of eq. (2), at each given order in perturbation theory, we express d arbitrary functions of N, $\gamma_k^{(p)}(N)$, in terms of the same number rational functions of N, $r_\ell^{(p)}(N)$. This has also an important technical implication, since in this way we are able to reconstruct the exact analytic form of the anomalous dimensions from solutions which are necessarily numerical for a large number of operators. On the other hand, the universality property of eq. (1), combined with the factorization properties in eqs. (40), (41) implies that the functions $w_\ell(g^2)$ are universal for all the operators which belong to a family of similar supermultiplets (with the same quantum numbers of the naive lowest component and the same number of components). In other words the naive scale dimension Δ_0, the spin s and the $SU(4)$ representation of the lowest component determine the whole set of functions $\{w_\ell(g^2)\}$. Hence to compute the perturbative anomalous dimension of any operator in the theory, $\gamma_k(g^2)$, in principle it is sufficient to specify its one loop anomalous dimension, $\gamma_k^{(1)}$, and the family of similar supermultiplets to which the operator belongs.

This suggests that it should be possible to obtain along these lines general formulae for the anomalous dimensions in $\mathcal{N}=4$ SYM theory. As an illustration that, if it exists, such a representation is far from obvious we shall write down the polynomial form of the order g^4 anomalous dimensions of the scalar operators of naive scale dimension $\Delta_0 = 4$ in the $\mathbf{20'}$ representation of the $SU(4)$ R-symmetry. There are four superprimary operators of this kind [23]. One is protected, while the order g^4 anomalous dimensions of the other three are given by[8]

$$\gamma_k^{(2)} = \frac{N^2}{(4\pi^2)^2}\frac{1}{4(N^6+116N^4-1180N^2+800)}\Big[(N^6-188N^4+1596N^2-1040)\eta_k^2 - 2(5N^6-133N^4+532N^2-220)\eta_k + (3N^2-2)(3N^4-112N^2+500)\Big]\,, \quad (50)$$

where $k=1,2,3$ and $\eta_k = \gamma_k^{(1)}/(\frac{N}{4\pi^2})$ are the roots of the cubic equation

$$N^2\eta^3 - 8\,N^2\eta^2 + 10(2\,N^2-1)\eta - 5(3\,N^2-2) = 0\,. \quad (51)$$

The details of the calculation, as well as the expressions for the normalizations of the 3-point functions and the OPE coefficients involving these operators will be presented elsewhere [40].

Acknowledgements

It is a pleasure to thank M. D'Alessandro, M. Bianchi and especially G. C. Rossi for numerous discussions and for comments on the manuscript. This work was supported in

[8]These anomalous dimensions, in a different but equivalent form, have been first computed in [30].

part by INFN, by the MIUR-COFIN contract 2003-023852, by the EU contracts MRTN-CT-2004-503369 and MRTN-CT-2004-512194, by the INTAS contract 03-51-6346, and by the NATO grant PST.CLG.978785.

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